Error Propagation in

Aided Discrete 2D Accelerometry

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Abstract

Closed form solutions for open and closed loop error propagation are available in the form of the convolution integrals and factorization solutions to the Riccati equation respectively. However, these are often not very illuminating unless the integrals and sums are actually carried out and simplified.

This report sets out to formulate and validate explicit models of systematic and stochastic error propagation in “accelerometry” - the author’s term for inertial navigation when the influence of gravity can be neglected. Under the assumption that the trajectory is a straight line, it turns out that the solution can be computed in closed form. Furthermore, when terrain relative velocity indications and measurements of heading (derived perhaps from a magnetometer) are available and integrated with a Kalman filter, it is possible to show in closed form their dramatic effect on overall system performance.

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1. Introduction

This document provides a rudimentary analysis of the error propagation for a simplified inertial navigation system consisting of a single forward accelerometer and a single vertical gyro. Such a system could be used to compute the position of a rolling wheel, for example, that operates in a horizontal plane. It was used to approximate the behavior of a Kalman filter estimating the motion of a foot walking on level ground.

The term accelerometry is used by analogy to odometry because the system will be assumed to be gravity compensated and hence not subject to Schuler dynamics. In other words, it will be assumed that the accelerometer readings have gravity removed. This is of course, straightforward if the system operates in a perfectly horizontal plane as is assumed here. In practice, the influence of gravity can be removed by frequent zero velocity updates which permit the explicit measurement and removal of the gravity indication in addition to the biases which would normally be computed anyway.

2. The Discrete-Time Linear System

Often, a system needs to be expressed in a discrete-time form in order to represent it in a computer. Sometimes the state equations are given in discrete form and other times they are generated by discretizing a continuous system.

2.1 Linear State Equations

If we are interested in a discrete-time representation, then the values of the vectors and matrices are known only at discrete times and the state equations take the form.

$$\begin{aligned} \underline{x}_{k+1} &= F_k \underline{x}_k + G_k \underline{u}_k \\ \underline{z}_k &= H_k \underline{x}_k + M_k \underline{u}_k \end{aligned} \qquad \textbf{(1)}$$

Here, the equations have similar form and similar meaning to the continuous case - with one exception. Note that $F(t)$ maps a state onto a state derivative while F_k maps a state onto a state. Also, whereas the continuous-time equations are differential equations, the discrete-time equations are recurrence equations.

2.1.1 Solution to the Linear State Equations

The solution to the state recurrence equations can be easily discovered by inspection by writing out

the terms as k increases from 0 to some general value n and noticing the pattern.

$$\underline{x}_1 = F_0\underline{x}_0 + G_0\underline{u}_0$$

$$\underline{x}_2 = F_1\underline{x}_1 + G_1\underline{u}_1$$

$$\underline{x}_3 = F_2\underline{x}_2 + G_2\underline{u}_2$$

$$\underline{x}_4 = F_3\underline{x}_3 + G_3\underline{u}_3$$

Unwinding the recursion:

$$\underline{x}_1 = F_0\underline{x}_0 + G_0\underline{u}_0$$

$$\underline{x}_2 = F_1[F_0\underline{x}_0 + G_0\underline{u}_0] + G_1\underline{u}_1$$

$$\underline{x}_3 = F_2[F_1[F_0\underline{x}_0 + G_0\underline{u}_0] + G_1\underline{u}_1] + G_2\underline{u}_2$$

$$\underline{x}_4 = F_3[F_2[F_1[F_0\underline{x}_0 + G_0\underline{u}_0] + G_1\underline{u}_1] + G_2\underline{u}_2] + G_3\underline{u}_3$$

The result of this tedious but straightforward exercise is:

$$x_n = \left(\prod_{k=0}^{n-1} F_k\right)x_0 + \sum_{k=0}^{n-1}\left[\prod_{p=k+1}^{n-1} F_p\right]G_k u_k \qquad \textbf{(2)}$$

By analogy to continuous-time, the discrete-time transition matrix is:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} F_p \qquad \textbf{(3)}$$

The product $\prod_{p=n}^{n-1} F_p$ is understood to mean[1] $F_{n-1}F_{n-2}\cdots F_k$:

- Both extremes of the indices appear in the product.

1. This convention is taken from Brogan page 220.

- When the high and low indices are the same, $\prod_{p=k}^{k} F_p = F_k$.
- When the interval is null, then $\prod_{p=n}^{n-1} F_p = 1$.

So the solution to the discrete differential equation can be written as:

$$x_n = \Phi_{n,0} x_0 + \sum_{k=0}^{n-1} \Phi_{n,k+1} G_k u_k \tag{4}$$

2.1.2 Solution for Commutable Dynamics

It is always possible to rewrite the system dynamics matrix as follows:

$$F_k = I + R_k \tag{5}$$

by simply solving for R_k.

Suppose that R_k can be partitioned as follows:

$$R_k = \left[\begin{array}{c|c} 0 & M_k \\ [n \times n] & [n \times m] \\ \hline 0 & 0 \\ [m \times n] & [m \times m] \end{array}\right] \tag{6}$$

or such matrices, it is easy to show that all cross products of R_k vanish. In particular:.

$$R_k R_{k+1} = 0$$

Under these conditions[1]:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} (I + R_p) = (I + R_k)(I + R_{k+1}) \ldots = I + \sum_{p=k}^{n-1} R_p \tag{7}$$

and we have converted a product into a sum as a result. Let this special transition matrix and sum

1. By convention $\sum_{p=n}^{n-1} R_p = 0$ hence $\Phi_{n,n} = I$ as expected.

be denoted as follows:

$$T_{n,k} = I + R_{n,k} = I + \sum_{p=k}^{n-1} R_p \qquad \textbf{(8)}$$

2.1.3 Solution with an Observer and Estimator

Consider now the case where there are measurements and an estimation system is used to refine estimates of the state. Let the aiding measurements be of the simple linear form:

$$\underline{z}_k = H_k \underline{x}_k \qquad \textbf{(9)}$$

These measurements may disagree with those computed from the dynamics so some mechanism to combine the two is necessary. In anticipation of later results, consider the use of a linear combination of the residual difference between the predicted measurements and the measurements:

$$\underline{x}_k^+ = \underline{x}_k^- + K_k(\underline{z}_k - H_k \underline{x}_k^-) = [I - K_k H_k]\underline{x}_k^- + K_k \underline{z}_k$$

Substituting this into the dynamics leads to:

$$\underline{x}_{k+1} = F_k([I - K_k H_k]\underline{x}_k + K_k \underline{z}_k) + G_k \underline{u}_k$$

Or

$$\underline{x}_{k+1} = F_k[I - K_k H_k]\underline{x}_k + F_k K_k \underline{z}_k + G_k \underline{u}_k \qquad \textbf{(10)}$$

This is now of the same form as the original system where the new system dynamics matrix is $F_k[I - K_k H_k]$ and the new inputs are both $F_k K_k \underline{z}_k$ and $G_k \underline{u}_k$. The solution to the state recurrence equations can be easily discovered by inspection by writing out the terms as k increases from 0 to some general value n and noticing the pattern:

$$\underline{x}_1 = F_0[I - K_0 H_0]\underline{x}_0 + F_0 K_0 \underline{z}_0 + G_0 \underline{u}_0$$

$$\underline{x}_2 = F_1[I - K_1 H_1]\underline{x}_1 + F_1 K_1 \underline{z}_1 + G_1 \underline{u}_1$$

$$\underline{x}_3 = F_2[I - K_2 H_2]\underline{x}_2 + F_2 K_2 \underline{z}_2 + G_2 \underline{u}_2$$

Unwinding the recursion for the last result:

$$\underline{x}_2 = F_1[I - K_1H_1]F_0[I - K_0H_0]\underline{x}_0 + F_1[I - K_1H_1][F_0K_0\underline{z}_0 + G_0\underline{u}_0] + [F_1K_1\underline{z}_1 + G_1\underline{u}_1]$$

$$\underline{x}_3 = F_2[I - K_2H_2]F_1[I - K_1H_1]F_0[I - K_0H_0]\underline{x}_0 + F_2[I - K_2H_2]F_1[I - K_1H_1][F_0K_0\underline{z}_0 + G_0\underline{u}_0] + F_2[I - K_2H_2][F_1K_1\underline{z}_1 + G_1\underline{u}_1] + F_2K_2\underline{z}_2 + G_2\underline{u}_2$$

The result of this tedious but straightforward exercise is:

$$x_n = \left(\prod_{k=0}^{n-1} F_k[I - K_kH_k] \right) x_0 + \sum_{k=0}^{n-1} \left[\prod_{p=k+1}^{n-1} F_p[I - K_pH_p] \right] \{ F_kK_k\underline{z}_k + G_ku_k \}$$

This is a dynamic system driven with two inputs. One is G_ku_k and the other is $F_kK_k\underline{z}_k$. By analogy to continuous-time, the discrete-time transition matrix is:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} F_p[I - K_pH_p] = \prod_{p=k}^{n-1} V_p \qquad \textbf{(11)}$$

Inspiring from equation (4), the solution to the discrete differential equation can be written as:

$$x_n = \Phi_{n,0}x_0 + \sum_{k=0}^{n-1} \Phi_{n,k+1}[F_kK_k\underline{z}_k + G_ku_k] \qquad \textbf{(12)}$$

2.1.4 Solution of Observed System with Special[1] Dynamics

Consider the case when the factors V_p in equation (11) are of the form:

$$V_p = (I + R_p)D_p = D_p + R_pD_p$$

Where R has the same form as equation (6) and D is a diagonal matrix with the special structure below:

$$D_k = \begin{bmatrix} I_{[n \times n]} & 0_{[n \times m]} \\ 0_{[m \times n]} & U_{k\,[m \times m]} \end{bmatrix} \qquad R_k = \begin{bmatrix} 0_{[n \times n]} & M_{k\,[n \times m]} \\ 0_{[m \times n]} & 0_{[m \times m]} \end{bmatrix} \qquad \textbf{(13)}$$

1. I'll use "special" until I can figure out a better name.

Note that under these conditions:

$$R_l R_k = 0 \quad \text{for all l,k} \qquad D_l R_k = R_k \quad \text{for all l,k}$$

This also means that:

$$V_l V_k = (R_l D_l + D_l)(R_k D_k + D_k) = R_l D_l R_k D_k + R_l D_l D_k + D_l R_k D_k + D_l D_k$$

But:

$$R_l D_l R_k D_k = R_l R_k D_k = 0$$

And:

$$D_l R_k D_k = R_k D_k$$

So:

$$V_l V_k = R_l D_l D_k + R_k D_k + D_l D_k$$

For such matrices, the transition matrix can be simplified. Consider the first few products:

$$V_{k+1} V_k = R_{k+1} D_{k+1} D_k + R_k D_k + D_{k+1} D_k$$

$$V_{k+2} V_{k+1} V_k = (R_{k+2} D_{k+2} + D_{k+2})(R_{k+1} D_{k+1} D_k + R_k D_k + D_{k+1} D_k) =$$
$$R_{k+2} D_{k+2} D_{k+1} D_k + R_{k+1} D_{k+1} D_k + R_k D_k + D_{k+2} D_{k+1} D_k$$

$$V_{k+3} V_{k+2} V_{k+1} V_k =$$
$$(R_{k+3} D_{k+3} + D_{k+3})(R_{k+2} D_{k+2} D_{k+1} D_k + R_{k+1} D_{k+1} D_k + R_k D_k + D_{k+2} D_{k+1} D_k) =$$
$$R_{k+3} D_{k+3} D_{k+2} D_{k+1} D_k + R_{k+2} D_{k+2} D_{k+1} D_k + R_{k+1} D_{k+1} D_k + R_k D_k + D_{k+3} D_{k+2} D_{k+1} D_k$$

The general pattern is therefore:

$$\prod_{p=k}^{n-1} V_p = \prod_{p=k}^{n-1} (I + R_p) D_p = \sum_{p=k}^{n-1} R_p \left(\prod_{i=k}^{p} D_i \right) + \prod_{p=k}^{n-1} D_p \qquad \textbf{(14)}$$

Define:

$$W_{n,k} = \prod_{p=k}^{n-1} D_p$$

Then the transition matrix is:

2.2 The Discrete-Time Nonlinear System and its Linear Perturbation

Nonlinear discrete-time systems are similar to their continuous-time counterparts.

$$\Phi_{n,k} = \sum_{p=k}^{n-1} R_p W_{p+1,k} + W_{n,k} \qquad \textbf{(15)}$$

2.2.1 Nonlinear State Equations

The nonlinear form of the state equations is:

$$\underline{x}_{k+1} = \underline{f}(\underline{x}_k, \underline{u}_k, k)$$
$$\underline{z}_k = \underline{h}(\underline{x}_k, k)$$

Even though a closed-form result for the nonlinear case may not be available, numerical solutions are available by direct recurrence on the first equation:.

$$\underline{x}_1 = \underline{f}(\underline{x}_0, \underline{u}_0, 0)$$
$$\underline{x}_2 = \underline{f}(\underline{x}_1, \underline{u}_1, 1)$$
$$\underline{x}_3 = \underline{f}(\underline{x}_2, \underline{u}_2, 2)$$
$$\ldots$$

2.2.2 Perturbation Theory

We can also model the behavior of a small "perturbation" about a known solution to the discrete-time state equations. Assume that a nominal input $\underline{u}_k$ and the associated nominal solution $\underline{x}_k$ are known. That is, they satisfy:

$$\underline{x}_{k+1} = \underline{f}(\underline{x}_k, \underline{u}_k, k) \qquad \textbf{(16)}$$

Suppose now that solution is desired for a slightly different input.

$$\underline{u}'_k = \underline{u}_k + \delta\underline{u}_k$$

Designate the solution associated with this input as follows:

$$\underline{x}'_k = \underline{x}_k + \delta\underline{x}_k$$

The state perturbation is again the difference between the perturbed and nominal state. This slightly

different solution, by definition, also satisfies the original state equation, so we can write:

$$\underline{x}'_{k+1} = \underline{x}_{k+1} + \delta\underline{x}_{k+1} = \underline{f}(\underline{x}_k + \delta\underline{x}_k, \underline{u}_k + \delta\underline{u}_k, k)$$

An approximation for $\delta\underline{x}_k$ will generate an approximation for $\underline{x}'_k$. We can get this approximation from the Taylor series expansion as follows:

$$\underline{f}(\underline{x}_k + \delta\underline{x}_k, \underline{u}_k + \delta\underline{u}_k, k) \approx \underline{f}(\underline{x}_k, \underline{u}_k, k) + F_k\delta\underline{x}_k + G_k\delta\underline{u}_k$$

where the two new matrices are the Jacobians of $\underline{f}$ with respect to the state and input - evaluated on the nominal trajectory:

$$F_k = \left.\frac{\partial}{\partial\underline{x}}\underline{f}\right|_{\underline{x}} \qquad G_k = \left.\frac{\partial}{\partial\underline{u}}\underline{f}\right|_{\underline{x}}$$

At this point, we have:

$$\underline{x}_{k+1} + \delta\underline{x}_{k+1} = \underline{f}(\underline{x}_k, \underline{u}_k, k) + F_k\delta\underline{x}_k + G_k\delta\underline{u}_k$$

Finally, by cancelling out the original state equation (16), there results a linear system which approximates the behavior of the perturbation.

$$\delta\underline{x}_{k+1} = F_k\delta\underline{x}_k + G_k\delta\underline{u}_k$$

All of the solution techniques for linear systems can now be applied to determine the behavior of this perturbation. Similar transformations can be used to linearize a nonlinear measurement equation to produce:

$$\delta\underline{z}_{k+1} = H_k\delta\underline{x}_k + M_k\delta\underline{u}_k$$

3. Error Propagation

This applies the results of the last one to produce general solutions for deterministic and stochastic error dynamics of linear and nonlinear dynamic systems.

3.1 Systematic Error Propagation for the Discrete-Time NonLinear System

Since equation (4) provides the solution for a linear system and a linearized (perturbed) system is linear. If the perturbations are interpreted as errors, we immediately have the solution for systematic error propagation:

$$\delta\underline{x}_n = \Phi_{n,0}\delta\underline{x}_0 + \sum_{k=0}^{n-1} \Phi_{n,k+1} G_k \delta\underline{u}_k = \Phi_{n,0}\delta\underline{x}_0 + \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1}\delta\underline{u}_k \qquad \textbf{(17)}$$

Where we have defined the “input transition matrix”:

$$\tilde{\Phi}_{n,k} = \Phi_{n,k} G_k \qquad \textbf{(18)}$$

And the transition matrix is:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} F_p \qquad \textbf{(19)}$$

3.2 Stochastic Error Propagation for the Discrete-Time NonLinear System

Since the above error propagation formula is linear in the variables of interest and since the state covariance is

$$P_n = Exp[\delta\underline{x}_n \delta\underline{x}_n^T] \qquad \textbf{(20)}$$

we have immediately:

$$P_n = \Phi_{n,0} P_0 \Phi_{n,0}^T + \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1} Q_k \tilde{\Phi}_{n,k+1}^T \qquad \textbf{(21)}$$

where, by definition:

$$Q_k = Exp[\delta\underline{u}_k \delta\underline{u}_k^T] \qquad \textbf{(22)}$$

3.3 Systematic Error Propagation With an Observer/Estimator

In the case of an observed system, we have from equation (12):

$$\delta x_n = \Phi_{n,0}\delta x_0 + \sum_{k=0}^{n-1} \Phi_{n,k+1}[F_k K_k \delta z_k + G_k \delta u_k] \tag{23}$$

$$\delta x_n = \Phi_{n,0}\delta x_0 + \sum_{k=0}^{n-1} \hat{\Phi}_{n,k+1}\delta z_k + \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1}\delta u_k$$

Where for the extra term, we have defined the “measurement transition matrix”:

$$\hat{\Phi}_{n,k} = \Phi_{n,k} F_k K_k \tag{24}$$

And the transition matrix is:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} F_p[I - K_p H_p] \tag{25}$$

3.4 Stochastic Error Propagation With an Observer/Estimator

Again since the above relationship is linear, we have immediately:

$$P_n = \Phi_{n,0} P_0 \Phi_{n,0}^T + \sum_{k=0}^{n-1} \hat{\Phi}_{n,k+1} S_k \hat{\Phi}_{n,k+1}^T + \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1} Q_k \tilde{\Phi}_{n,k+1}^T \tag{26}$$

Also, by definition:

$$S_k = Exp[\delta \underline{z}_k \delta \underline{z}_k^T] \tag{27}$$

4. Single Wheel Accelerometry

Consider the problem of "accelerometry" in the plane where acceleration is integrated twice and angular velocity is integrated once. Let the state vector include position and orientation and linear velocity. The system is driven by measurements of angular velocity about the vertical and acceleration in the forward direction.

The state equations are:

$$\begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 & 0 & c\theta\Delta t \\ 0 & 1 & 0 & s\theta\Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_k \begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_k + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \omega \\ a \end{bmatrix}_k \tag{28}$$

4.1 Linearization

This is a nonlinear system, so we linearize it as follows:

$$\delta \underline{x}_{k+1} = F_k \delta \underline{x}_k + G_k \delta \underline{u}_k$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 1 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_k \begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_k + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \delta\omega \\ \delta a \end{bmatrix}_k \tag{29}$$

4.2 Transition Matrix

The transition matrix is:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} F_p \tag{30}$$

Note that:

$$F_p = I + R_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 0 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_p \tag{31}$$

And especially note that $R_p R_{p+1} = 0$. Hence, we have based on equation (7):

$$\Phi_{n,k} = I + \sum_{p=k}^{n-1} R_p \tag{32}$$

Now, define:

$$R_{n,k} = \sum_{p=k}^{n-1} R_p = \sum_{p=k}^{n-1} \begin{bmatrix} 0 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 0 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 & -\Delta y_{n,k} & C_{n,k} \\ 0 & 0 & \Delta x_{n,k} & S_{n,k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

Where:

$$\Delta x_{n,k} = \sum_{p=k}^{n-1} Vc\theta\Delta t = \sum_{p=k}^{n-1} c\theta\Delta s = x_n - x_k$$

$$\Delta y_{n,k} = \sum_{p=k}^{n-1} Vs\theta\Delta t = \sum_{p=k}^{n-1} s\theta\Delta s = y_n - y_k \tag{34}$$

$$C_{n,k} = \sum_{p=k}^{n-1} c\theta\Delta t \qquad S_{n,k} = \sum_{p=k}^{n-1} s\theta\Delta t$$

The transition matrix therefore is:

$$\Phi_{n,k} = \begin{bmatrix} 1 & 0 & -\Delta y_{n,k} & C_{n,k} \\ 0 & 1 & \Delta x_{n,k} & S_{n,k} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

Define the “input transition matrix”:

$$\tilde{\Phi}_{n,k} = \Phi_{n,k} G_k = \begin{bmatrix} 1 & 0 & -\Delta y_{n,k} & C_{n,k} \\ 0 & 1 & \Delta x_{n,k} & S_{n,k} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} = \Delta t \begin{bmatrix} -\Delta y_{n,k} & C_{n,k} \\ \Delta x_{n,k} & S_{n,k} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{36}$$

4.3 Systematic Error Propagation

The solution for systematic error propagation is:

$$\delta\underline{x}_n = \Phi_{n,0}\delta\underline{x}_0 + \sum_{k=0}^{n-1} \Phi_{n,k+1} G_k \delta\underline{u}_k = \Phi_{n,0}\delta\underline{x}_0 + \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1}\delta\underline{u}_k$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = \begin{bmatrix} 1 & 0 & -\Delta y_{n,0} & C_{n,k+1} \\ 0 & 1 & \Delta x_{n,0} & S_{n,k+1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_0 + \sum_{k=0}^{n-1} \Delta t \begin{bmatrix} -\Delta y_{n,k+1} & C_{n,k+1} \\ \Delta x_{n,k+1} & S_{n,k+1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\omega \\ \delta a \end{bmatrix}_k$$

For vanishing initial error, this is:

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = \sum_{k=0}^{n-1} \Delta t \begin{bmatrix} -\Delta y_{n,k+1} & C_{n,k+1} \\ \Delta x_{n,k+1} & S_{n,k+1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\omega \\ \delta a \end{bmatrix}_k$$

For constant Δt and constant error magnitudes, this is:

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = \Delta t \sum_{k=0}^{n-1} \begin{bmatrix} -\delta\omega\Delta y_{n,k+1} + \delta a C_{n,k+1} \\ \delta\omega\Delta x_{n,k+1} + \delta a S_{n,k+1} \\ \delta\omega \\ \delta a \end{bmatrix}$$

4.3.1 Straight Line Trajectory

On a straight trajectory along the x axis, we have $\Delta y_{n,k+1} = 0$, $C_{n,k+1} = (n-(k+1))\Delta t$, $\Delta x_{n,k+1} = x_n - x_{k+1}$, $S_{n,k+1} = 0$. Hence, we have:

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = \Delta t \sum_{k=0}^{n-1} \begin{bmatrix} \Delta t\delta a(n-(k+1)) \\ \delta\omega(x_n - x_{k+1}) \\ \delta\omega \\ \delta a \end{bmatrix}$$

The first line is:

$$\delta x = \Delta t^2 \delta a([n-1] + [n-2] + \ldots + [n-n]) = \Delta t^2 \delta a \sum_{k=0}^{n-1} k$$

The sum can be simplified using the identity:

$$\sum_{k=0}^{n} i = n(n+1)/2$$

Therefore, the first line simplifies to:

$$\boxed{\delta x = \Delta t^2 \delta a \frac{[n(n-1)]}{2} \approx \frac{\Delta t^2 \delta a[n^2]}{2}} \quad \textbf{(37)}$$

The second line is:

$$\delta y = \Delta t \delta\omega([x_n - x_1] + [x_n - x_2] + \ldots + [x_n - x_{n-1}])$$

For constant velocity, we have:

$$x_k = k\Delta x = kV\Delta t$$

Giving:

$$\delta y = \Delta t^2 V \delta\omega([n-1] + [n-2] + \ldots + [n-(n-1)]) = \Delta t^2 V \delta\omega \sum_{k=0}^{n-1} k$$

Using our finite sum identity again leads to:

$$\boxed{\delta y = \Delta t^2 V \delta\omega \frac{[n(n-1)]}{2} \approx \Delta t^2 V \delta\omega \frac{[n^2]}{2}} \quad \textbf{(38)}$$

Hence, the complete result is:

$$\boxed{\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = \Delta t \sum_{k=0}^{n-1} \begin{bmatrix} \Delta t \delta a(n-1-k) \\ \delta\omega(x_{n-1} - x_k) \\ \delta\omega \\ \delta a \end{bmatrix} = \Delta t \begin{bmatrix} \Delta t \delta a[n^2]/2 \\ \Delta t V \delta\omega[n^2]/2 \\ \delta\omega[n] \\ \delta a[n] \end{bmatrix} = \begin{bmatrix} \delta a t^2/2 \\ V\delta\omega t^2/2 \\ \delta\omega t \\ \delta a t \end{bmatrix}} \quad \textbf{(39)}$$

Of course, $n\Delta t = t$ so systematic position error becomes quadratic in time whereas heading and velocity error are linear.

4.4 Stochastic Error Propagation

Once again, the systematic error propagation formula is:

$$\delta\underline{x}_n = \Phi_{n,0}\delta\underline{x}_0 + \sum_{k=0}^{n-1}\Phi_{n,k+1}G_k\delta\underline{u}_k = \sum_{k=0}^{n-1}\tilde{\Phi}_{n,k+1}\delta\underline{u}_k$$

Since this is linear in the variables of interest and since the state covariance is

$$P_n = Exp[\delta\underline{x}_n\delta\underline{x}_n^T]$$

we have immediately:

$$P_n = \Phi_{n,0}P_0\Phi_{n,0}^T + \sum_{k=0}^{n-1}\tilde{\Phi}_{n,k+1}Q_k\tilde{\Phi}_{n,k+1}^T$$

where, by definition:

$$Q_k = Exp[\delta\underline{u}_k\delta\underline{u}_k^T]$$

This is:

$$\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xV}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{yV}\\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta V}\\ \sigma_{Vx} & \sigma_{Vy} & \sigma_{V\theta} & \sigma_{VV}\end{bmatrix}_n = \begin{bmatrix}1 & 0 & -\Delta y_{n,0} & C_{n,0}\\ 0 & 1 & \Delta x_{n,0} & S_{n,0}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xV}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{yV}\\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta V}\\ \sigma_{Vx} & \sigma_{Vy} & \sigma_{V\theta} & \sigma_{VV}\end{bmatrix}_0\begin{bmatrix}1 & 0 & -\Delta y_{n,0} & C_{n,0}\\ 0 & 1 & \Delta x_{n,0} & S_{n,0}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}^T$$
$$+\sum_{k=0}^{n-1}\Delta t\begin{bmatrix}-\Delta y_{n,k+1} & C_{n,k+1}\\ \Delta x_{n,k+1} & S_{n,k+1}\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_{\omega\omega} & \sigma_{\omega a}\\ \sigma_{a\omega} & \sigma_{aa}\end{bmatrix}_k\Delta t\begin{bmatrix}-\Delta y_{n,k+1} & C_{n,k+1}\\ \Delta x_{n,k+1} & S_{n,k+1}\\ 1 & 0\\ 0 & 1\end{bmatrix}^T$$

For vanishing initial error, this is:

$$\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xV}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{yV}\\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta V}\\ \sigma_{Vx} & \sigma_{Vy} & \sigma_{V\theta} & \sigma_{VV}\end{bmatrix}_n = \sum_{k=0}^{n-1}\Delta t\begin{bmatrix}-\Delta y_{n,k+1} & C_{n,k+1}\\ \Delta x_{n,k+1} & S_{n,k+1}\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_{\omega\omega} & \sigma_{\omega a}\\ \sigma_{a\omega} & \sigma_{aa}\end{bmatrix}_k\Delta t\begin{bmatrix}-\Delta y_{n,k+1} & C_{n,k+1}\\ \Delta x_{n,k+1} & S_{n,k+1}\\ 1 & 0\\ 0 & 1\end{bmatrix}^T$$

For constant Δt and constant error magnitudes, this is:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xV} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{yV} \\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta V} \\ \sigma_{Vx} & \sigma_{Vy} & \sigma_{V\theta} & \sigma_{VV} \end{bmatrix}_n = \Delta t^2 \sum_{k=0}^{n-1} \begin{bmatrix} -\Delta y_{n,k+1} & C_{n,k+1} \\ \Delta x_{n,k+1} & S_{n,k+1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\omega\omega} & \sigma_{\omega a} \\ \sigma_{a\omega} & \sigma_{aa} \end{bmatrix}_k \begin{bmatrix} -\Delta y_{n,k+1} & C_{n,k+1} \\ \Delta x_{n,k+1} & S_{n,k+1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T$$

4.4.1 Straight Line Trajectory

On a straight trajectory along the x axis, we have $\Delta y_{n,k+1} = 0$, $S_{n,k} = 0$. Let us also assume that the accelerometer and gyro errors are decorrelated so that $\sigma_{a\omega} = \sigma_{\omega a} = 0$. Hence, we have:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xV} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{yV} \\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta V} \\ \sigma_{Vx} & \sigma_{Vy} & \sigma_{V\theta} & \sigma_{VV} \end{bmatrix}_n = \Delta t^2 \sum_{k=0}^{n-1} \begin{bmatrix} 0 & C_{n,k+1} \\ \Delta x_{n,k+1} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\omega\omega} & 0 \\ 0 & \sigma_{aa} \end{bmatrix}_k \begin{bmatrix} 0 & C_{n,k+1} \\ \Delta x_{n,k+1} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xV} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{yV} \\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta V} \\ \sigma_{Vx} & \sigma_{Vy} & \sigma_{V\theta} & \sigma_{VV} \end{bmatrix}_n = \Delta t^2 \sum_{k=0}^{n-1} \begin{bmatrix} 0 & \sigma_{aa} C_{n,k+1} \\ \sigma_{\omega\omega} \Delta x_{n,k+1} & 0 \\ \sigma_{\omega\omega} & 0 \\ 0 & \sigma_{aa} \end{bmatrix} \begin{bmatrix} 0 & \Delta x_{n,k+1} & 1 & 0 \\ C_{n,k+1} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xV} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{yV} \\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta V} \\ \sigma_{Vx} & \sigma_{Vy} & \sigma_{V\theta} & \sigma_{VV} \end{bmatrix}_n = \Delta t^2 \sum_{k=0}^{n-1} \begin{bmatrix} \sigma_{aa} C_{n,k+1}^2 & 0 & 0 & \sigma_{aa} C_{n,k+1} \\ 0 & \sigma_{\omega\omega} \Delta x_{n,k+1}^2 & \sigma_{\omega\omega} \Delta x_{n,k+1} & 0 \\ 0 & \sigma_{\omega\omega} \Delta x_{n,k+1} & \sigma_{\omega\omega} & 0 \\ \sigma_{aa} C_{n,k+1} & 0 & 0 & \sigma_{aa} \end{bmatrix}$$

Concentrating on the diagonal (variances) of this expression, we have:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{\theta\theta} \\ \sigma_{VV} \end{bmatrix}_n = \Delta t^2 \sum_{k=0}^{n-1} \begin{bmatrix} \sigma_{aa} C_{n,k+1}^2 \\ \sigma_{\omega\omega} \Delta x_{n,k+1}^2 \\ \sigma_{\omega\omega} \\ \sigma_{aa} \end{bmatrix}$$

On a straight trajectory along the x axis, we also have $C_{n,k+1} = (n-(k+1))\Delta t$, $\Delta x_{n,k+1} = x_n - x_{k+1}$.

The first line is:

$$\sigma_{xx} = \Delta t^4 \sigma_{aa}([n-1]^2 + [n-2]^2 + \ldots + [n-n]^2) = \Delta t^4 \sigma_{aa} \sum_{k=0}^{n-1} k^2$$

The sum can be simplified using the identity:

$$\sum_{k=1}^{n-1} i^2 = \sum_{k=0}^{n-1} i^2 = n(n-1)(2n-1)/6$$

Therefore, the first line simplifies to:

$$\sigma_{xx} = \Delta t^4 \sigma_{aa} \frac{[n(n-1)(2n-1)]}{6} = \Delta t^4 \sigma_{aa} \frac{n[2n^2 - 3n + 1]}{6} \approx \frac{\Delta t^4 \sigma_{aa}[n^3]}{3}$$

The second line is:

$$\sigma_{yy} = \Delta t^2 \sigma_{\omega\omega}([x_n - x_1]^2 + [x_n - x_2]^2 + \ldots + [x_n - x_n]^2)$$

For constant velocity, we have:

$$x_k = k\Delta x = kV\Delta t$$

Giving:

$$\sigma_{yy} = \Delta t^4 V^2 \sigma_{\omega\omega}([n-1]^2 + [n-2]^2 + \ldots + [n-n]^2) = \Delta t^4 V^2 \sigma_{\omega\omega} \sum_{k=0}^{n-1} k^2$$

Using our finite sum identity again leads to:

$$\sigma_{yy} = \Delta t^4 V^2 \sigma_{\omega\omega} \frac{n[2n^2 - 3n + 1]}{6} \approx \Delta t^4 V^2 \sigma_{\omega\omega} \frac{[n^3]}{3}$$

Hence, the final result for variance is:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{\theta\theta} \\ \sigma_{VV} \end{bmatrix}_n = \Delta t^2 \begin{bmatrix} \Delta t^2 \sigma_{aa} \frac{[n^3]}{3} \\ \Delta t^2 V^2 \sigma_{\omega\omega} \frac{[n^3]}{3} \\ \sigma_{\omega\omega}[n] \\ \sigma_{aa}[n] \end{bmatrix} = \Delta t \begin{bmatrix} \sigma_{aa} \frac{t^3}{3} \\ V^2 \sigma_{\omega\omega} \frac{t^3}{3} \\ \sigma_{\omega\omega} t \\ \sigma_{aa} t \end{bmatrix} \quad \textbf{(40)}$$

Of course, $n\Delta t = t$ so stochastic position error (variance) becomes cubic in time whereas heading and velocity error (variance) are linear.

5. Aided Single Wheel Accelerometry

Consider now the case where the accelerometry system is aided by measurements of heading and velocity.

The state equations are:

$$\begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 & 0 & c\theta\Delta t \\ 0 & 1 & 0 & s\theta\Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_k \begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_k + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \omega \\ a \end{bmatrix}_k$$

The measurement relationship is:

$$\underline{z}_k = H_k \underline{x}_k$$

$$\begin{bmatrix} z_\theta \\ z_V \end{bmatrix}_k = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_k \begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_k \qquad \textbf{(41)}$$

Let the estimator relationship be as follows:

$$\underline{x}_k^+ = \underline{x}_k^- + K_k(\underline{z}_k - H_k \underline{x}_k^-)$$

$$\begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_k^+ = \begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_k^- + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_\theta & 0 \\ 0 & k_V \end{bmatrix} \left\{ \begin{bmatrix} z_\theta \\ z_V \end{bmatrix}_k - \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_k \begin{bmatrix} x \\ y \\ \theta \\ V \end{bmatrix}_k^- \right\} \qquad \textbf{(42)}$$

We would normally choose both k_θ and k_V to be less than unity.

5.1 Linearization

The linearized system dynamics is as before:

$$\delta\underline{x}_{k+1} = F_k \delta\underline{x}_k + G_k \delta\underline{u}_k$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 1 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_k \begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_k + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \delta\omega \\ \delta a \end{bmatrix}_k$$

5.2 Transition Matrix

The transition matrix is:

$$\Phi_{n,\,k} = \prod_{p\,=\,k}^{n\,-\,1} F_p[I - K_pH_p] = \prod_{p\,=\,k}^{n\,-\,1} V_p$$

Note that:

$$F_p = I + R_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 0 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_p$$

$$K_pH_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_\theta & 0 \\ 0 & k_V \end{bmatrix}_p \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_\theta & 0 \\ 0 & 0 & 0 & k_V \end{bmatrix}$$

Notice that because K_pH_p is diagonal, $[I - K_pH_p]$ has special diagonal structure:

$$F_p[I - K_pH_p] = F_pD_p = (I + R_p)D_p$$

Where:

$$R_p = \begin{bmatrix} 0 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 0 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad D_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1 - k_\theta) & 0 \\ 0 & 0 & 0 & (1 - k_V) \end{bmatrix}$$

And, in this case, V_p is structured as in equation (13):

$$R_k = \left[\begin{array}{c|c} \underset{[n \times n]}{0} & \underset{[n \times m]}{M_k} \\ \hline \underset{[m \times n]}{0} & \underset{[m \times m]}{0} \end{array}\right] \qquad D_k = \left[\begin{array}{c|c} \underset{[n \times n]}{I} & \underset{[n \times m]}{0} \\ \hline \underset{[m \times n]}{0} & \underset{[m \times m]}{U_k} \end{array}\right] \qquad \textbf{(43)}$$

So, the transition matrix is given by equation (15):

$$\Phi_{n,k} = \prod_{p=k}^{n-1} V_p = \prod_{p=k}^{n-1} (I + R_p)D_p = \sum_{p=k}^{n-1} R_p W_{p+1,k} + W_{n,k} \qquad \textbf{(44)}$$

Where:

$$W_{n,k} = \prod_{p=k}^{n-1} D_p$$

The second part of equation (44) reduces to:

$$W_{n,k} = \prod_{p=k}^{n-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1-k_\theta) & 0 \\ 0 & 0 & 0 & (1-k_V) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-k_\theta) & 0 \\ 0 & 0 & 0 & (1-k_V) \end{bmatrix}^{n-k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-k_\theta)^{n-k} & 0 \\ 0 & 0 & 0 & (1-k_V)^{n-k} \end{bmatrix}$$

The second part of equation (44) reduces to:

$$\sum_{p=k}^{n-1} R_p W_{p+1,k} = \sum_{p=k}^{n-1} \begin{bmatrix} 0 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 0 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-k_\theta)^{(p+1)-k} & 0 \\ 0 & 0 & 0 & (1-k_V)^{(p+1)-k} \end{bmatrix}$$

$$\sum_{p=k}^{n-1} R_p W_{p+1,k} = \sum_{p=k}^{n-1} \begin{bmatrix} 0 & 0 & -Vs\theta\Delta t(1-k_\theta)^{(p+1)-k} & c\theta\Delta t(1-k_V)^{(p+1)-k} \\ 0 & 0 & Vc\theta\Delta t(1-k_\theta)^{(p+1)-k} & s\theta\Delta t(1-k_V)^{(p+1)-k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the transition matrix is:

$$\Phi_{n,k} = \sum_{p=k}^{n-1} R_p W_{p+1,k} + W_{n,k}$$

$$\Phi_{n,k} = \sum_{p=k}^{n-1} \begin{bmatrix} 0 & 0 & -Vs\theta\Delta t(1-k_\theta)^{(p+1)-k} & c\theta\Delta t(1-k_V)^{(p+1)-k} \\ 0 & 0 & Vc\theta\Delta t(1-k_\theta)^{(p+1)-k} & s\theta\Delta t(1-k_V)^{(p+1)-k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-k_\theta)^{n-k} & 0 \\ 0 & 0 & 0 & (1-k_V)^{n-k} \end{bmatrix}$$

$$\Phi_{n,k} = \begin{bmatrix} 0 & 0 & -\Delta\tilde{y}_{n,k} & \tilde{C}_{n,k} \\ 0 & 0 & \Delta\tilde{x}_{n,k} & \tilde{S}_{n,k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k_\theta^{n,k} & 0 \\ 0 & 0 & 0 & k_V^{n,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta\tilde{y}_{n,k} & \tilde{C}_{n,k} \\ 0 & 1 & \Delta\tilde{x}_{n,k} & \tilde{S}_{n,k} \\ 0 & 0 & k_\theta^{n,k} & 0 \\ 0 & 0 & 0 & k_V^{n,k} \end{bmatrix}$$

Where, analogously to the unobserved case:

$$\Delta\tilde{x}_{n,k} = \sum_{p=k}^{n-1} Vc\theta\Delta t(1-k_\theta)^{(p+1)-k} = \sum_{p=k}^{n-1} c\theta\Delta s k_\theta^{p+1,k} \qquad k_\theta^{n,k} = (1-k_\theta)^{n-k}$$

$$\Delta\tilde{y}_{n,k} = \sum_{p=k}^{n-1} Vs\theta\Delta t(1-k_\theta)^{(p+1)-k} = \sum_{p=k}^{n-1} s\theta\Delta s k_\theta^{p+1,k} \qquad \textbf{(45)}$$

$$\tilde{C}_{n,k} = \sum_{p=k}^{n-1} c\theta\Delta t k_V^{p+1,k} \qquad \tilde{S}_{n,k} = \sum_{p=k}^{n-1} s\theta\Delta t k_V^{p+1,k} \qquad k_V^{n,k} = (1-k_V)^{n-k}$$

The coefficients $k_\theta^{n,k}$ and $k_V^{n,k}$ decrease rapidly toward zero as the exponent $n-k$ increases and as the gains approach unity.

Define the "input transition matrix":

$$\tilde{\Phi}_{n,k} = \Phi_{n,k} G_k = \begin{bmatrix} 1 & 0 & -\Delta\tilde{y}_{n,k} & \tilde{C}_{n,k} \\ 0 & 1 & \Delta\tilde{x}_{n,k} & \tilde{S}_{n,k} \\ 0 & 0 & k_\theta^{n,k} & 0 \\ 0 & 0 & 0 & k_V^{n,k} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} = \Delta t \begin{bmatrix} -\Delta\tilde{y}_{n,k} & \tilde{C}_{n,k} \\ \Delta\tilde{x}_{n,k} & \tilde{S}_{n,k} \\ k_\theta^{n,k} & 0 \\ 0 & k_V^{n,k} \end{bmatrix}$$

In this case:

$$F_p K_p = \begin{bmatrix} 1 & 0 & -Vs\theta\Delta t & c\theta\Delta t \\ 0 & 1 & Vc\theta\Delta t & s\theta\Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_\theta & 0 \\ 0 & k_V \end{bmatrix}_p = \begin{bmatrix} -k_\theta Vs\theta\Delta t & k_V c\theta\Delta t \\ k_\theta Vc\theta\Delta t & k_V s\theta\Delta t \\ k_\theta & 0 \\ 0 & k_V \end{bmatrix}$$

Using this, define the "measurement transition matrix":

$$\hat{\Phi}_{n,k} = \Phi_{n,k} F_k K_k = \begin{bmatrix} 1 & 0 & -\Delta\tilde{y}_{n,k} & \tilde{C}_{n,k} \\ 0 & 1 & \Delta\tilde{x}_{n,k} & \tilde{S}_{n,k} \\ 0 & 0 & k_\theta^{n,k} & 0 \\ 0 & 0 & 0 & k_V^{n,k} \end{bmatrix} \begin{bmatrix} -k_\theta Vs\theta\Delta t & k_V c\theta\Delta t \\ k_\theta Vc\theta\Delta t & k_V s\theta\Delta t \\ k_\theta & 0 \\ 0 & k_V \end{bmatrix}$$

$$\hat{\Phi}_{n,k} = \begin{bmatrix} -k_\theta Vs\theta\Delta t - k_\theta\Delta\tilde{y}_{n,k} & k_V c\theta\Delta t + k_V\tilde{C}_{n,k} \\ k_\theta Vc\theta\Delta t + k_\theta\Delta\tilde{x}_{n,k} & k_V s\theta\Delta t + k_V\tilde{S}_{n,k} \\ k_\theta k_\theta^{n,k} & 0 \\ 0 & k_V k_V^{n,k} \end{bmatrix}$$

5.3 Systematic Error Propagation

The solution for systematic error propagation for vanishing initial error is by equation (23):

$$\delta x_n = \sum_{k=0}^{n-1} \hat{\Phi}_{n,k+1}\delta z_k + \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1}\delta u_k$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = \sum_{k=0}^{n-1} \left[\begin{array}{cc} -k_\theta Vs\theta\Delta t - k_\theta\Delta\tilde{y}_{n,k+1} & k_V c\theta\Delta t + k_V\tilde{C}_{n,k+1} \\ k_\theta Vc\theta\Delta t + k_\theta\Delta\tilde{x}_{n,k+1} & k_V s\theta\Delta t + k_V\tilde{S}_{n,k+1} \\ k_\theta k_\theta^{n,k+1} & 0 \\ 0 & k_V k_V^{n,k+1} \end{array}\right] \begin{bmatrix} \delta z_\theta \\ \delta z_V \end{bmatrix}_k + \sum_{k=0}^{n-1} \Delta t \begin{bmatrix} -\Delta\tilde{y}_{n,k+1} & \tilde{C}_{n,k+1} \\ \Delta\tilde{x}_{n,k+1} & \tilde{S}_{n,k+1} \\ k_\theta^{n,k+1} & 0 \\ 0 & k_V^{n,k+1} \end{bmatrix} \begin{bmatrix} \delta\omega \\ \delta a \end{bmatrix}_k$$

This can be written as:

$$\begin{bmatrix}\delta x\\ \delta y\\ \delta\theta\\ \delta V\end{bmatrix}_n = \sum_{k=0}^{n-1} k_\theta(\delta z_\theta)_k \begin{bmatrix}-Vs\theta\Delta t - \Delta\tilde{y}_{n,k+1}\\ Vc\theta\Delta t + \Delta\tilde{x}_{n,k+1}\\ k_\theta^{n,k+1}\\ 0\end{bmatrix} + \sum_{k=0}^{n-1} k_V(\delta z_V)_k \begin{bmatrix}c\theta\Delta t + \tilde{C}_{n,k+1}\\ s\theta\Delta t + \tilde{S}_{n,k+1}\\ 0\\ k_V^{n,k+1}\end{bmatrix} +$$

$$\sum_{k=0}^{n-1} \Delta t(\delta\omega)_k \begin{bmatrix}-\Delta\tilde{y}_{n,k+1}\\ \Delta\tilde{x}_{n,k+1}\\ k_\theta^{n,k+1}\\ 0\end{bmatrix} + \sum_{k=0}^{n-1} \Delta t(\delta a)_k \begin{bmatrix}\tilde{C}_{n,k+1}\\ \tilde{S}_{n,k+1}\\ 0\\ k_V^{n,k+1}\end{bmatrix}$$

Which simplifies to:

$$\begin{bmatrix}\delta x\\ \delta y\\ \delta\theta\\ \delta V\end{bmatrix}_n = \sum_{k=0}^{n-1} \{k_\theta(\delta z_\theta)_k + \Delta t(\delta\omega)_k\} \begin{bmatrix}-\Delta\tilde{y}_{n,k+1}\\ \Delta\tilde{x}_{n,k+1}\\ k_\theta^{n,k+1}\\ 0\end{bmatrix} + \sum_{k=0}^{n-1} \{k_V(\delta z_V)_k + \Delta t(\delta a)_k\} \begin{bmatrix}\tilde{C}_{n,k+1}\\ \tilde{S}_{n,k+1}\\ 0\\ k_V^{n,k+1}\end{bmatrix} +$$

$$\sum_{k=0}^{n-1} k_\theta(\delta z_\theta)_k \begin{bmatrix}-Vs\theta\Delta t\\ Vc\theta\Delta t\\ 0\\ 0\end{bmatrix} + \sum_{k=0}^{n-1} k_V(\delta z_V)_k \begin{bmatrix}c\theta\Delta t\\ s\theta\Delta t\\ 0\\ 0\end{bmatrix}$$

5.3.1 Straight Line trajectory

On a straight trajectory along the x axis, we have $\Delta\tilde{y}_{n,k+1} = 0$ and $\tilde{S}_{n,k+1} = 0$.

Also:

$$\tilde{C}_{n,k+1} = \sum_{p=k+1}^{n-1} c\theta\Delta t k_V^{p+1,k+1} = \Delta t \sum_{p=k+1}^{n-1} k_V^{p+1,k+1} = \Delta t \tilde{k}_V^{n,k+1} \qquad \textbf{(46)}$$

Where, we have defined:

$$\tilde{k}_V^{n,k} = \sum_{p=k}^{n-1} k_V^{p,k} = \sum_{p=k}^{n-1} (1-k_V)^{p-k} \qquad \textbf{(47)}$$

And, for constant velocity on a straight line:

$$\Delta\tilde{x}_{n,\,k+1} = \sum_{p\,=\,k+1}^{n-1} c\theta\Delta s k_\theta^{p+1,\,k+1} = \Delta s \sum_{p\,=\,k+1}^{n-1} k_\theta^{p+1,\,k+1} = \Delta s \tilde{k}_\theta^{n,\,k+1} \qquad \textbf{(48)}$$

Where we have defined:

$$\tilde{k}_\theta^{n,\,k} = \sum_{p\,=\,k}^{n-1} k_\theta^{p,\,k} = \sum_{p\,=\,k}^{n-1} (1 - k_\theta)^{p-k} \qquad \textbf{(49)}$$

For constant error magnitudes on a straight line, the above result is:

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = (\delta z_\theta k_\theta + \delta\omega\Delta t) \sum_{k\,=\,0}^{n-1} \begin{bmatrix} 0 \\ \Delta s \tilde{k}_\theta^{n,\,k+1} \\ k_\theta^{n,\,k+1} \\ 0 \end{bmatrix} + (\delta z_V k_V + \delta a\Delta t) \sum_{k\,=\,0}^{n-1} \begin{bmatrix} \Delta t \tilde{k}_V^{n,\,k+1} \\ 0 \\ 0 \\ k_V^{n,\,k+1} \end{bmatrix} + \sum_{k\,=\,0}^{n-1} \begin{bmatrix} \delta z_V k_V \Delta t \\ \delta z_\theta k_\theta V \Delta t \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{(50)}$$

Define $\beta_x = n_x \Delta t$ as the first line inside the second sum in equation (50):

$$\beta_x = \tilde{C}_{n,\,k+1} = \Delta t \tilde{k}_V^{n,\,k+1} = \Delta t \left\{ \sum_{p\,=\,k+1}^{n-1} k_V^{p+1,\,k+1} \right\} = \Delta t \left\{ \sum_{p\,=\,k+1}^{n-1} (1 - k_V)^{p-k} \right\}$$

Then n_x is:

$$n_x = \sum_{p\,=\,k+1}^{n-1} (1 - k_V)^{p-k} = \sum_{p\,=\,0}^{n-k-2} (1 - k_V)^{p+1} = (1 - k_V)^{n-k-1} + (1 - k_V)^{n-k-2} + \ldots + (1 - k_V)$$

The rising powers of $(1 - k_V) \ll 1$ quickly cause n_x to converge. When n is large relative to k, n_x is related to an essentially infinite geometric series which converges to:

$$n_x \sim \left\{ \sum_{p\,=\,0}^{\infty} (1 - k_V)^p \right\} - 1 = \frac{1}{k_V} - 1 = \left(\frac{1 - k_V}{k_V} \right) \qquad \textbf{(51)}$$

Some values are tabulated below:

Table 1:

k_V	$1/k_V$	n_x
0.9	1.11	0.11
0.8	1.25	0.25
0.7	1.43	0.43

Table 1:

k_V	$1/k_V$	n_x
0.6	1.67	0.67
0.5	2.0	1.00

After only a few cycles n will be large enough that the following formula holds:

$$\beta_x = n_x \Delta t = \Delta t \left(\frac{1 - k_V}{k_V} \right)$$

The summation of all these is:

$$\gamma_x = \sum_{k=0}^{n-1} \tilde{C}_{n,\,k+1} = \sum_{k=0}^{n-1} \Delta t \left(\frac{1 - k_V}{k_V} \right) = t_n \left(\frac{1 - k_V}{k_V} \right)$$

Define $\beta_y = n_y \Delta s$ as the second line inside the first sum in equation (50):

$$\beta_y = \Delta \tilde{x}_{n,\,k+1} = \Delta s \tilde{k}_\theta^{n,\,k+1} = \Delta s \left\{ \sum_{p=k+1}^{n-1} k_\theta^{p+1,\,k+1} \right\} = \Delta s \left\{ \sum_{p=k+1}^{n-1} (1 - k_\theta)^{p-k} \right\}$$

Then n_y is

$$n_y = \sum_{p=k+1}^{n-1} (1 - k_\theta)^{p-k} = \sum_{p=0}^{n-k-2} (1 - k_\theta)^{p+1} = (1 - k_\theta)^{n-k-1} + (1 - k_\theta)^{n-k-2} + \ldots + (1 - k_\theta)$$

Proceeding as for n_x

$$n_y \sim \left\{ \sum_{p=0}^{\infty} (1 - k_\theta)^p \right\} - 1 = \frac{1}{k_\theta} - 1 = \left(\frac{1 - k_\theta}{k_\theta} \right) \qquad \textbf{(52)}$$

After only a few cycles n is large enough that the following formula holds:

$$\beta_y = n_y \Delta s = \Delta s \left(\frac{1 - k_\theta}{k_\theta} \right)$$

The summation of all these is:

$$\gamma_y = \sum_{k=0}^{n-1} \Delta \tilde{x}_{n,\,k} = \sum_{k=0}^{n-1} \Delta s \left(\frac{1 - k_\theta}{k_\theta} \right) = x_n \left(\frac{1 - k_\theta}{k_\theta} \right)$$

Define n_θ as the third line of the first sum in equation (50):

$$n_\theta = \sum_{k=0}^{n-1} k_\theta^{n,k+1} = \sum_{k=0}^{n-1} (1-k_\theta)^{n-(k+1)} = (1-k_\theta)^{n-1} + (1-k_\theta)^{n-2} + \ldots + 1$$

This is essentially the same sum as for n_y only longer and the unity term is present. Hence for large n:

$$n_\theta = \sum_{k=0}^{n-1} (1-k_\theta)^{n-k} \sim \left(\frac{1}{k_\theta}\right) \qquad \textbf{(53)}$$

Define n_V as the fourth line of the second sum:

$$n_V = \sum_{k=0}^{n-1} k_V^{n,k+1} = \sum_{k=0}^{n-1} (1-k_V)^{n-(k+1)} = (1-k_V)^{n-1} + (1-k_V)^{n-2} + \ldots + 1$$

This is essentially the same sum as for n_x only longer and the unity term is present. Hence for large n:

$$n_V = \sum_{k=0}^{n-1} (1-k_V)^{n-k} \sim \left(\frac{1}{k_V}\right) \qquad \textbf{(54)}$$

Consider the first line of the third sum in equation (50):

$$\kappa_x = \sum_{k=0}^{n-1} \delta z_V k_V \Delta t = \delta z_V k_V \sum_{k=0}^{n-1} \Delta t = \delta z_V k_V t_n$$

Consider the second line of the third sum in equation (50):

$$\kappa_y = \sum_{k=0}^{n-1} \delta z_\theta k_\theta V \Delta t = \delta z_\theta k_\theta \sum_{k=0}^{n-1} V \Delta t = \delta z_\theta k_\theta x_n$$

The total solution is therefore:

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \\ \delta V \end{bmatrix}_n = (\delta z_\theta k_\theta + \delta\omega\Delta t) \begin{bmatrix} 0 \\ x_n\left(\frac{1-k_\theta}{k_\theta}\right) \\ 1/k_\theta \\ 0 \end{bmatrix} + (\delta z_V k_V + \delta a \Delta t) \begin{bmatrix} t_n\left(\frac{1-k_V}{k_V}\right) \\ 0 \\ 0 \\ 1/k_V \end{bmatrix} + \begin{bmatrix} \delta z_V k_V t_n \\ \delta z_\theta k_\theta x_n \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{(55)}$$

Which simplifies to:

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta \theta \\ \delta V \end{bmatrix}_n = \begin{bmatrix} \delta z_V t_n + \delta a \Delta t t_n (1 - k_V / k_V) \\ \delta z_\theta x_n + \delta \omega \Delta t x_n (1 - k_\theta / k_\theta) \\ \delta z_\theta + \delta \omega \Delta t / k_\theta \\ \delta z_V + \delta a \Delta t / k_V \end{bmatrix} \quad \textbf{(56)}$$

5.4 Tolerable Error Magnitudes

Systematic position error is linear in time and distance whereas heading and velocity error are constants (because they are being measured directly on a regular basis).

If we allocate 1/4 of a position error budget to each sensor along its sensitive axis, we can write relationships for computing the sensor errors from the specifications.

$$\begin{bmatrix} \delta x/4 \\ \delta y/4 \end{bmatrix} = \begin{bmatrix} \delta z_V t_n \\ \delta z_\theta x_n \end{bmatrix} \qquad \begin{bmatrix} \delta x/4 \\ \delta y/4 \end{bmatrix} = \begin{bmatrix} \delta a \Delta t t_n (1 - k_V / k_V) \\ \delta \omega \Delta t x_n (1 - k_\theta / k_\theta) \end{bmatrix}$$

Solving leads to:

$$\begin{bmatrix} \delta z_\theta \\ \delta z_V \\ \delta \omega \\ \delta a \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \delta y / x_n \\ \delta x / t_n \\ \delta y / (\Delta t x_n (1 - k_\theta / k_\theta)) \\ \delta x / (\Delta t t_n (1 - k_V / k_V)) \end{bmatrix} \quad \textbf{(57)}$$

5.5 Stochastic Error Propagation

Once again, the systematic error propagation formula is by (26):

$$P_n = \Phi_{n,0} P_0 \Phi_{n,0}^T + \sum_{k=0}^{n-1} \hat{\Phi}_{n,k+1} S_k \hat{\Phi}_{n,k+1}^T + \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1} Q_k \tilde{\Phi}_{n,k+1}^T \quad \textbf{(58)}$$

Where the state covariance is

$$P_n = \mathrm{Exp}[\delta \underline{x}_n \delta \underline{x}_n^T]$$

And the input and state covariances are definition:

$$S_k = \mathrm{Exp}[\delta \underline{z}_k \delta \underline{z}_k^T] \qquad Q_k = \mathrm{Exp}[\delta \underline{u}_k \delta \underline{u}_k^T]$$

Lets assume vanishing initial error and simplify the other two terms.

5.5.1 Measurement Contribution

The contribution of the measurements is:

$$P_{z_n} = \sum_{k=0}^{n-1} \hat{\Phi}_{n,k+1} S_k \hat{\Phi}^T_{n,k+1} =$$

$$\sum_{k=0}^{n-1} \begin{bmatrix} -k_\theta V s\theta \Delta t - k_\theta \Delta\tilde{y}_{n,k+1} & k_V c\theta \Delta t + k_V \tilde{C}_{n,k+1} \\ k_\theta V c\theta \Delta t + k_\theta \Delta\tilde{x}_{n,k+1} & k_V s\theta \Delta t + k_V \tilde{S}_{n,k+1} \\ k_\theta k_\theta^{n,k+1} & 0 \\ 0 & k_V k_V^{n,k+1} \end{bmatrix} \begin{bmatrix} \sigma_{z_\theta z_\theta} & \sigma_{z_\theta z_V} \\ \sigma_{z_V z_\theta} & \sigma_{z_V z_V} \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}^T$$

For assumed decorrelated measurements ($\sigma_{z_\theta z_V} = 0$) and concentrating on the diagonal (variances) in this expression we have:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{\theta\theta} \\ \sigma_{VV} \end{bmatrix}_{z_n} = \sum_{k=0}^{n-1} \begin{bmatrix} \sigma_{z_\theta z_\theta}(-k_\theta V s\theta \Delta t - k_\theta \Delta\tilde{y}_{n,k+1})^2 + \sigma_{z_V z_V}(k_V c\theta \Delta t + k_V \tilde{C}_{n,k+1})^2 \\ \sigma_{z_\theta z_\theta}(k_\theta V c\theta \Delta t + k_\theta \Delta\tilde{x}_{n,k+1})^2 + \sigma_{z_V z_V}(k_V s\theta \Delta t + k_V \tilde{S}_{n,k+1})^2 \\ \sigma_{z_\theta z_\theta}(k_\theta k_\theta^{n,k+1})^2 \\ \sigma_{z_V z_V}(k_V k_V^{n,k+1})^2 \end{bmatrix}$$

5.5.2 Measurement Contribution on a Straight Line Trajectory

On a straight trajectory along the x axis, we have $\Delta\tilde{y}_{n,k+1} = 0$ and $\tilde{S}_{n,k+1} = 0$ etc.

Hence, this simplifies to:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{\theta\theta} \\ \sigma_{VV} \end{bmatrix}_{z_n} = \sum_{k=0}^{n-1} \begin{bmatrix} \sigma_{z_V z_V}(k_V \Delta t + k_V \tilde{C}_{n,k+1})^2 \\ \sigma_{z_\theta z_\theta}(k_\theta V \Delta t + k_\theta \Delta\tilde{x}_{n,k+1})^2 \\ \sigma_{z_\theta z_\theta}(k_\theta k_\theta^{n,k+1})^2 \\ \sigma_{z_V z_V}(k_V k_V^{n,k+1})^2 \end{bmatrix}$$

Also, by equations (46) (47) (48) (49) on this trajectory for constant velocity:

$$\tilde{C}_{n,k+1} = \Delta t \sum_{p=k+1}^{n-1} k_V^{p+1,k+1} = \Delta t \tilde{k}_V^{n,k+1} \qquad \tilde{k}_V^{n,k} = \sum_{p=k}^{n-1} k_V^{p,k} = \sum_{p=k}^{n-1} (1-k_V)^{p-k}$$
$$\Delta \tilde{x}_{n,k+1} = \Delta s \sum_{p=k+1}^{n-1} k_\theta^{p+1,k+1} = \Delta s \tilde{k}_\theta^{n,k+1} \qquad \tilde{k}_\theta^{n,k} = \sum_{p=k}^{n-1} k_\theta^{p,k} = \sum_{p=k}^{n-1} (1-k_\theta)^{p-k} \qquad \textbf{(59)}$$

So the result is:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{\theta\theta} \\ \sigma_{VV} \end{bmatrix}_{z_n} = \sum_{k=0}^{n-1} \begin{bmatrix} \sigma_{z_V z_V} (k_V \Delta t + k_V \Delta t \tilde{k}_V^{n,k+1})^2 \\ \sigma_{z_\theta z_\theta} (k_\theta V \Delta t + k_\theta \Delta s \tilde{k}_\theta^{n,k+1})^2 \\ \sigma_{z_\theta z_\theta} (k_\theta k_\theta^{n,k+1})^2 \\ \sigma_{z_V z_V} (k_V k_V^{n,k+1})^2 \end{bmatrix} = \begin{bmatrix} \sigma_{z_V z_V} (k_V \Delta t)^2 \sum_{k=0}^{n-1} (1 + \tilde{k}_V^{n,k+1})^2 \\ \sigma_{z_V z_V} (k_\theta \Delta s)^2 \sum_{k=0}^{n-1} (1 + \tilde{k}_\theta^{n,k+1})^2 \\ \sigma_{z_\theta z_\theta} (k_\theta)^2 \sum_{k=0}^{n-1} (k_\theta^{n,k+1})^2 \\ \sigma_{z_V z_V} (k_V)^2 \sum_{k=0}^{n-1} (k_V^{n,k+1})^2 \end{bmatrix}$$

Assuming n is large, we can simplify the first two lines using equations (51) and (52):

$$n_x = \tilde{k}_V^{n,k+1} \sim \frac{1}{k_V} - 1 \qquad n_y = \tilde{k}_\theta^{n,k+1} \sim \frac{1}{k_\theta} - 1 \qquad \textbf{(60)}$$

Now, we must define two new sums for the third and fourth lines. First, for the third line:

$$n_{\theta\theta} = \sum_{k=0}^{n-1} (k_\theta^{n,k+1})^2 = \sum_{k=0}^{n-1} (1-k_\theta)^{2n-2(k+1)} = (1-k_\theta)^{2(n-1)} + (1-k_\theta)^{2(n-2)} + \ldots + 1$$
$$n_{VV} = \sum_{k=0}^{n-1} (k_V^{n,k+1})^2 = \sum_{k=0}^{n-1} (1-k_V)^{2n-2(k+1)} = (1-k_V)^{2(n-1)} + (1-k_V)^{2(n-2)} + \ldots + 1$$

The rising powers of $(1-k_\theta) \ll 1$ quickly cause $n_{\theta\theta}$ to converge. When n is large relative to k, $n_{\theta\theta}$ is related to an essentially infinite geometric series. The same formula applies after reinterpreting the common factor, so this converges to:

$$n_{\theta\theta} = \sum_{k=0}^{n-1} (k_\theta^{n,k+1})^2 = \sum_{p=0}^{\infty} (1-k_\theta)^{2p} \sim \frac{1}{1-(1-k_\theta)^2} = \frac{1}{1-(1-2k_\theta+k_\theta^2)} = \frac{1}{k_\theta(2-k_\theta)} \qquad \textbf{(61)}$$

Similiarly we have:

$$n_{VV} = \sum_{k=0}^{n-1} \left(k_V^{n,k+1}\right)^2 = \sum_{p=0}^{\infty} (1-k_V)^{2p} \sim \frac{1}{1-(1-k_V)^2} = \frac{1}{1-(1-2k_V+k_V^2)} = \frac{1}{k_V(2-k_V)} \qquad \textbf{(62)}$$

So the result is:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{\theta\theta} \\ \sigma_{VV} \end{bmatrix}_{z_n} = \begin{bmatrix} \sigma_{z_V z_V}(k_V \Delta t)^2 \sum_{k=0}^{n-1} \left(\frac{1}{k_V}\right)^2 \\ \sigma_{z_V z_V}(k_\theta \Delta s)^2 \sum_{k=0}^{n-1} \left(\frac{1}{k_\theta}\right)^2 \\ \sigma_{z_\theta z_\theta}(k_\theta)^2 \frac{1}{k_\theta(2-k_\theta)} \\ \sigma_{z_V z_V}(k_V)^2 \frac{1}{k_V(2-k_V)} \end{bmatrix}$$

This simplifies to:

$$\boxed{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{\theta\theta} \\ \sigma_{VV} \end{bmatrix}_{z_n} = \begin{bmatrix} n\sigma_{z_V z_V}\Delta t^2 \\ n\sigma_{z_V z_V}\Delta s^2 \\ \sigma_{z_\theta z_\theta}(k_\theta/(2-k_\theta)) \\ \sigma_{z_V z_V}(k_V/(2-k_V)) \end{bmatrix}}$$

5.5.3 Input Contribution

The contribution of the inputs is:

$$P_{U_n} = \sum_{k=0}^{n-1} \tilde{\Phi}_{n,k+1} S_k \tilde{\Phi}_{n,k+1}^T = \sum_{k=0}^{n-1} \Delta t \begin{bmatrix} -\Delta\tilde{y}_{n,k+1} & \tilde{C}_{n,k+1} \\ \Delta\tilde{x}_{n,k+1} & \tilde{S}_{n,k+1} \\ k_\theta^{n,k+1} & 0 \\ 0 & k_V^{n,k+1} \end{bmatrix} \begin{bmatrix} \sigma_{\omega\omega} & \sigma_{\omega a} \\ \sigma_{a\omega} & \sigma_{aa} \end{bmatrix} \Delta t \begin{bmatrix} -\Delta\tilde{y}_{n,k+1} & \tilde{C}_{n,k+1} \\ \Delta\tilde{x}_{n,k+1} & \tilde{S}_{n,k+1} \\ k_\theta^{n,k+1} & 0 \\ 0 & k_V^{n,k+1} \end{bmatrix}^T$$

For assumed decorrelated measurements ($\sigma_{a\omega} = 0$) and concentrating on the diagonal (variances)

in this expression we have:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{\theta\theta}\\ \sigma_{VV}\end{bmatrix}_{u_n} = \sum_{k=0}^{n-1} \Delta t^2 \begin{bmatrix}\sigma_{\omega\omega}(-\Delta\tilde{y}_{n,k+1})^2 + \sigma_{aa}(\tilde{C}_{n,k+1})^2\\ \sigma_{\omega\omega}(\Delta\tilde{x}_{n,k+1})^2 + \sigma_{aa}(\tilde{S}_{n,k+1})^2\\ \sigma_{\omega\omega}(k_\theta^{n,k+1})^2\\ \sigma_{aa}(k_V^{n,k+1})^2\end{bmatrix}$$

5.5.4 Input Contribution on a Straight Line Trajectory

On a straight trajectory along the x axis, we have $\Delta\tilde{y}_{n,k+1} = 0$ and $\tilde{S}_{n,k+1} = 0$.
Hence, this simplifies to:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{\theta\theta}\\ \sigma_{VV}\end{bmatrix}_{u_n} = \sum_{k=0}^{n-1} \Delta t^2 \begin{bmatrix}\sigma_{aa}(\tilde{C}_{n,k+1})^2\\ \sigma_{\omega\omega}(\Delta\tilde{x}_{n,k+1})^2\\ \sigma_{\omega\omega}(k_\theta^{n,k+1})^2\\ \sigma_{aa}(k_V^{n,k+1})^2\end{bmatrix}$$

We can reuse the simplifications from (59) and (60) above which are:

$$\tilde{C}_{n,k+1} = \Delta t\tilde{k}_V^{n,k+1} \sim \Delta t\left(\frac{1}{k_V} - 1\right) \qquad \Delta\tilde{x}_{n,k+1} = \Delta s\tilde{k}_\theta^{n,k+1} \sim \Delta s\left(\frac{1}{k_\theta} - 1\right)$$

Also from (61) and (62):

$$n_{\theta\theta} = \sum_{k=0}^{n-1} (k_\theta^{n,k+1})^2 \sim \frac{1}{k_\theta(2-k_\theta)} \qquad n_{VV} = \sum_{k=0}^{n-1} (k_V^{n,k+1})^2 \sim \frac{1}{k_V(2-k_V)}$$

The result is now:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{\theta\theta}\\ \sigma_{VV}\end{bmatrix}_{u_n} = \sum_{k=0}^{n-1} \Delta t^2 \begin{bmatrix}\sigma_{aa}\Delta t^2\left(\frac{1}{k_V} - 1\right)^2\\ \sigma_{\omega\omega}\Delta s^2\left(\frac{1}{k_\theta} - 1\right)^2\\ \sigma_{\omega\omega}(k_\theta^{n,k+1})^2\\ \sigma_{aa}(k_V^{n,k+1})^2\end{bmatrix} = n\Delta t^2 \begin{bmatrix}\sigma_{aa}\Delta t^2((1-k_V)/k_V)^2\\ \sigma_{\omega\omega}\Delta s^2((1-k_\theta)/k_\theta)^2\\ \sigma_{\omega\omega}(1/(k_\theta(2-k_\theta)))\\ \sigma_{aa}(1/(k_V(2-k_V)))\end{bmatrix}$$

5.5.5 Total Error

The total error for vanishing initial conditions is the sum of the measurement and input contributions:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{\theta\theta}\\ \sigma_{VV}\end{bmatrix} = \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{\theta\theta}\\ \sigma_{VV}\end{bmatrix}_{z_n} + \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{\theta\theta}\\ \sigma_{VV}\end{bmatrix}_{u_n} = \begin{bmatrix} n\sigma_{z_V z_V}\Delta t^2 \\ n\sigma_{z_\theta z_\theta}\Delta s^2 \\ \sigma_{z_\theta z_\theta}(k_\theta/(2-k_\theta)) \\ \sigma_{z_V z_V}(k_V/(2-k_V))\end{bmatrix} + \Delta t^2\begin{bmatrix} n\sigma_{aa}\Delta t^2((1-k_V)/k_V)^2 \\ n\sigma_{\omega\omega}\Delta s^2((1-k_\theta)/k_\theta)^2 \\ \sigma_{\omega\omega}(1/(k_\theta(2-k_\theta))) \\ \sigma_{aa}(1/(k_V(2-k_V)))\end{bmatrix} \quad \textbf{(63)}$$

5.6 Tolerable Error Magnitudes

Stochastic position error (variance) becomes linear in time whereas heading and velocity error are constants (because they are being measured directly on a regular basis).

If we allocate 1/4 of a position error budget to each sensor along its sensitive axis, we can write relationships for computing the sensor errors from the specifications.

$$\begin{bmatrix}\sigma_{xx}/4\\ \sigma_{yy}/4\end{bmatrix} = \begin{bmatrix}\sigma_{z_V z_V}\Delta tt_n \\ \sigma_{z_\theta z_\theta}\Delta sx_n\end{bmatrix} \qquad \begin{bmatrix}\sigma_{xx}/4\\ \sigma_{yy}/4\end{bmatrix} = \Delta t^2\begin{bmatrix}\sigma_{aa}\Delta tt_n((1-k_V)/k_V)^2 \\ \sigma_{\omega\omega}\Delta sx_n((1-k_\theta)/k_\theta)^2\end{bmatrix}$$

Solving leads to:

$$\begin{bmatrix}\sigma_{z_\theta z_\theta}\\ \sigma_{z_V z_V}\\ \sigma_{\omega\omega}\\ \sigma_{aa}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}\sigma_{yy}/\Delta sx_n \\ \sigma_{xx}/\Delta tt_n \\ \sigma_{yy}/(\Delta t^2\Delta sx_n/((1-k_\theta)/k_\theta)^2) \\ \sigma_{xx}/(\Delta t^3 t_n/(1-k_V)/k_V{}^2)\end{bmatrix} \quad \textbf{(64)}$$

5.7 Validation

Equations (56) and (63) are the main results for this section. They can be used in the forms in equations (57) and (64) in order to design a system to meet a specification. Due to the “large n” approximations used, it seems prudent to validate these models against some real data. Two effects can be expected from the assumptions used to simplify the models. First, when n is not large - namely at the start of the system - the behavior may not be predicted exactly. Second, a small gains assumption was used in summing many geometric series, so the formulas are not valid as the gain approaches unity. Nor are they valid at zero gains due to division by zero. This latter zero gain (unaided) case was modelled independently in earlier chapters. Both of these limitations can be removed, at the cost of incerased complexity, by refining the formulas used for the coefficients n_x etc. which are used throughout the derivations.

By way of validation, a Kalman filter was constructed and provided with corrupted data according to a set of error specifications to compare how the error that occurs in practice compares with the error that is predicted by the models.

The results only apply to a straight line trajectory so such a trajectory was used. The distance between updates Δs is set to 1 meter. For an assumed velocity of 5 km/hr, this corresponds to a time period of 0.72 seconds between readings. The goal terminal position error was 10 meters over a period of 0.5 hours. Kalman gains k_θ and k_V were set to 0.1. This setting weights inertial sensing roughly ten times as highly as heading and velocity sensing in order to reduce requirements on velocity sensing. The following error magnitudes were used.

Table 2: Required Sensor Residual Biases And Standard Deviations

	Residual Bias δ Phase I	Residual Bias δ Phase II	Standard Deviation σ Phase I	Standard Deviation σ Phase II
Heading z_θ	0.052 deg	0.0033 deg	5.2 deg	0.9 deg
Velocity z_V	1.3 mm/s	0.1 mm/s	126 mm/s	22 mm/s
Angular Velocity ω	0.008 deg/ sec	0.0005 deg/ sec	0.8 deg/sec	0.15 deg/sec
Acceleration a	0.02 mg's	0.001 mg's	2 mg's	0.35 mg's

5.7.1 Systematic Error

The simulation of systematic errors is straightforward. The heading, velocity, gyro, and accelerometer inputs are corrupted by constant biases of the magnitudes given in Table 2 and the resulting trajectory is compared to the nominal one. Figure 1 provides the result for the systematic error model.

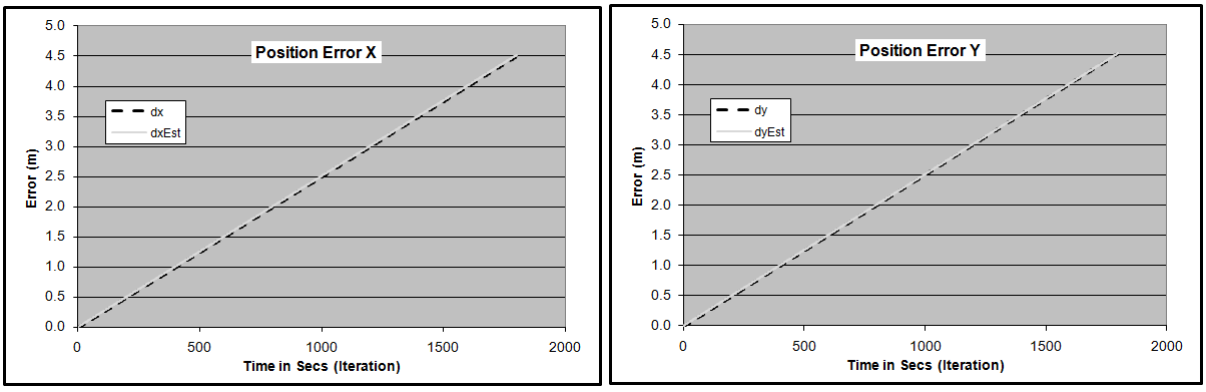


Figure 1 **Systematic Error Propagation Model Accuracy.** The systematic error model is a slight overestimate but it correctly models the linear growth of error with time.

The error model correctly captures the linear behavior with excellent precision.

5.7.2 Random Error

The simulation of random errors is not so straightforward. The heading, velocity, gyro, and accelerometer inputs were corrupted by (Gaussian) random perturbations of the magnitudes (variances) given in Table 2 for every cycle of the filter and the resulting trajectory is compared to the nominal one. This comparison was performed for 250 different simulations each comprised of 4 corrupted sensor signals provided every second for 1800 seconds. Figure 2 provides the result for the random error model.

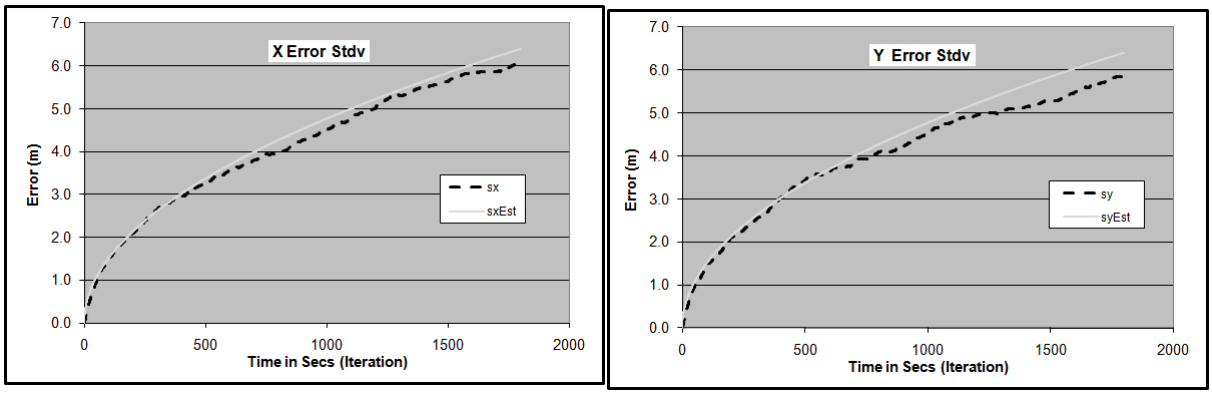


Figure 2 **Random Error Propagation Model Accuracy.** The random error model is a slight overestimate but it correctly models the square root growth of error with time.

Once again, the model correctly captures the square root growth of standard deviation with time. The agreement is excellent in the short term and slight deviations in the long term are likely due to insufficient (and computationally expensive to generate) Monte Carlo data.

Overall, however, both models are clearly more than adequate for predicting system performance and deriving sensor specification on systematic and random errors.